A Concurrency Problem with Exponential DPLL(\(\mathcal{T}\)) Proofs

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Outline

SAT/SMT-based Verification Techniques for Concurrency

DPLL($T$) Lower Bound Proof Complexity Theorem

A Concurrency Problem with $O(N!)$-sized DPLL($T$) Proofs
  Two State-of-the-art Partial-Order Encodings

Experiments

Concluding Remarks
SAT/SMT-based Verification Techniques

SAT/SMT solvers are highly optimized decision procedures:

SAT

\[ \phi \rightarrow \square \]

UNSAT

Using these, state-of-the-art symbolic bounded model checker such as CBMC can find concurrency bugs in Apache, PostgreSQL and others.
The Problem

Can we pinpoint the challenges (if any) symbolic partial-order encodings of concurrency pose for SAT/SMT solvers?

If yes, what insights can we gain?
SMT Solvers built on DPLL(T)

SAT\textsubscript{main} CDCL

assertions

Arrays

conflicts

propagations

explanations

UF

LRA

Bit-vectors

core

SAT\textsubscript{main} CDCL
Fixed-Alphabet DPLL(\(\mathcal{T}\)) Proofs

The SMT solvers in our experiments are built on the DPLL(\(\mathcal{T}\)) framework. A simplified form of DPLL(\(\mathcal{T}\)) with only two rules:

- Propositional resolution (Res);
- Learning \(\mathcal{T}\)-valid clauses over the literals of a fixed alphabet of \(\mathcal{T}\)-atoms (\(\mathcal{T}\)-learn).
Fixed-Alphabet DPLL(\(T\)) Proofs

The SMT solvers in our experiments are built on the DPLL(\(T\)) framework. A simplified form of DPLL(\(T\)) with only two rules:

- Propositional resolution (\(\text{Res}\));
- Learning \(T\)-valid clauses over the literals of a fixed alphabet of \(T\)-atoms (\(T\)-\text{learn}).

Example: \((x < y \lor x = y) \land y < x\) where \(x, y \in \mathbb{Z}\). This formula is \(T\)-unsatisfiable where \(T\) is QF\_LIA.
Fixed-Alphabet DPLL(T) Proofs

Example: \((x < y \lor x = y) \land y < x\)
Fixed-Alphabet DPLL($\mathcal{T}$) Proofs

Example: $(x < y \lor x = y) \land y < x$

$\sim (A \lor B) \land C$

(SAT: Yes)
Fixed-Alphabet DPLL(\(T\)) Proofs

Example: \((x < y \lor x = y) \land y < x\)

\[\sim \Rightarrow (A \lor B) \land C \land (\neg A \lor \neg C)\] (SAT: Yes)

\[\sim \Rightarrow \left(\begin{array}{c} A \lor B \end{array}\right) \land C \land \left(\begin{array}{c} \neg A \lor \neg C \end{array}\right)\] (T-LEARN)
Fixed-Alphabet DPLL($\mathcal{T}$) Proofs

Example: $(x < y \lor x = y) \land y < x$

\[
\sim (A \lor B) \land (\neg A \lor \neg C) \land (A \lor B) \\
\sim (A \lor B) \land (\neg A \lor \neg C) \land (A \lor B) \\
\sim (A \lor B) \land (\neg A \lor \neg C) \land (A \lor B) \\
\sim \bot
\]

(SAT: Yes)  
($\mathcal{T}$-LEARN)  
(SAT: Yes)  
($\mathcal{T}$-LEARN)  
(Res)
Known Challenges for SMT Solvers

\[
\phi \triangleq a_1 \neq a_9 \land \bigwedge_{i=1}^{8} (a_i = b_i \land b_i = a_{i+1}) \lor (a_i = c_i \land c_i = a_{i+1})
\]
Known Challenges for SMT Solvers

\[ \phi \equiv a_1 \neq a_9 \land \bigwedge_{i=1}^{8} (a_i = b_i \land b_i = a_{i+1}) \lor (a_i = c_i \land c_i = a_{i+1}) \]
Known Challenges for SMT Solvers

\[ \phi \triangleq a_1 \neq a_9 \land \bigwedge_{i=1}^{8} (a_i = b_i \land b_i = a_{i+1}) \lor (a_i = c_i \land c_i = a_{i+1}) \]
Known Challenges for SMT Solvers

\[ \phi = a_1 \neq a_9 \land \bigwedge_{i=1}^{8} (a_i = b_i \land b_i = a_{i+1}) \lor (a_i = c_i \land c_i = a_{i+1}) \]

\[
\begin{array}{c}
\text{a}_1 \\
\downarrow \\
\text{c}_1
\end{array}
\begin{array}{c}
\text{a}_2 \\
\downarrow \\
\text{c}_2
\end{array}
\begin{array}{c}
\text{a}_3 \\
\downarrow \\
\text{c}_3
\end{array}
\begin{array}{c}
\text{a}_4 \\
\ldots
\end{array}
\begin{array}{c}
\text{a}_8 \\
\downarrow \\
\text{c}_8 \\
\downarrow \\
\text{a}_9
\end{array}
\]

\[
\begin{array}{c}
\text{b}_1 \\
\downarrow \\
\text{a}_1
\end{array}
\begin{array}{c}
\text{b}_2 \\
\downarrow \\
\text{a}_2
\end{array}
\begin{array}{c}
\text{b}_3 \\
\downarrow \\
\text{a}_3
\end{array}
\begin{array}{c}
\text{b}_8 \\
\downarrow \\
\text{a}_8
\end{array}
\]

\[
\begin{array}{c}
\text{b}_1 \\
\downarrow \\
\text{a}_1
\end{array}
\begin{array}{c}
\text{b}_2 \\
\downarrow \\
\text{a}_2
\end{array}
\begin{array}{c}
\text{b}_3 \\
\downarrow \\
\text{a}_3
\end{array}
\begin{array}{c}
\text{b}_8 \\
\downarrow \\
\text{a}_8
\end{array}
\]

\[
\begin{array}{c}
\text{b}_1 \\
\downarrow \\
\text{a}_1
\end{array}
\begin{array}{c}
\text{b}_2 \\
\downarrow \\
\text{a}_2
\end{array}
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\text{b}_3 \\
\downarrow \\
\text{a}_3
\end{array}
\begin{array}{c}
\text{b}_8 \\
\downarrow \\
\text{a}_8
\end{array}
\]

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\begin{array}{c}
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\text{a}_1
\end{array}
\begin{array}{c}
\text{b}_2 \\
\downarrow \\
\text{a}_2
\end{array}
\begin{array}{c}
\text{b}_3 \\
\downarrow \\
\text{a}_3
\end{array}
\begin{array}{c}
\text{b}_8 \\
\downarrow \\
\text{a}_8
\end{array}
\]
Known Challenges for SMT Solvers

\[ \phi_3 \triangleq a_1 \neq a_9 \land \bigwedge_{i=1}^{8} (a_i = b_i \land b_i = a_{i+1}) \lor (a_i = c_i \land c_i = a_{i+1}) \]

- Note that \( \phi_3 \) is unsatisfiable because \( a_i = a_{i+1} \) for all \( 1 \leq i \leq 8 \).
- But DPLL(\( T \)) cannot learn these equalities and enumerates \( 2^8 \) theory conflicts instead, the infamous \textit{diamonds problem}.\(^1\)

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\(^1\)Bjørner, N., Dutertre, B., de Moura, L.: \textit{Accelerating Lemma Learning using Joins - DPLL(Join)}. In: LPAR (2008)
Contributions

- A general framework for establishing lower bounds on the number of $\mathcal{T}$-conflicts in the fixed-alphabet DPLL($\mathcal{T}$) calculus;
- Proof of factorial lower bound proof complexity for two state-of-the-art symbolic partial-order encodings of a simple, yet challenging concurrency problem;
- Experiments that confirm our theoretical lower bound.

\[
\Omega(N!)
\]

\[
\Omega(2^N)
\]
DPLL(\(\mathcal{T}\)) Lower Bound Proof Complexity

Informally, *non-interfering set of critical assignments* are propositionally satisfying assignments that contain minimal, disjoint \(\mathcal{T}\)-conflicts.

**Theorem**

Let \(\phi\) be an unsatisfiable \(\mathcal{T}\)-formula, and \(Q\) be a non-interfering set of critical assignments for \(\phi\). Every Fixed-Alphabet-DPLL(\(\mathcal{T}\)) proof that \(\phi\) is UNSAT contains at least \(|Q|\) applications of \(\mathcal{T}\)-\textsc{learn}.

This theorem is a theoretical tool for proving lower bound proof complexity results in the DPLL(\(\mathcal{T}\)) framework (as exhibited next).
The Concurrency Problem

The value at memory location \( x \) is initialized to zero, i.e. \( [x] = 0 \).

<table>
<thead>
<tr>
<th>Thread ( T_0 )</th>
<th>Thread ( T_1 )</th>
<th>Thread ( T_N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>\textbf{local} ( v_0 := [x] )</td>
<td>\textbf{local} ( v_1 := [x] )</td>
<td>\textbf{local} ( v_N := [x] )</td>
</tr>
<tr>
<td>\textbf{assert}(v_0 \leq N)</td>
<td>\textbf{[x]} := v_1 + 1</td>
<td>\textbf{[x]} := v_N + 1</td>
</tr>
</tbody>
</table>

For \( N = 2 \), if restricted to \( T_1 \parallel T_2 \), we get the following interleavings:

1. \( r_1; w_1; r_2; w_2 \)  
2. \( r_1; r_2; w_1; w_2 \)  
3. \( r_1; r_2; w_2; w_1 \)  
4. \( r_2; r_1; w_1; w_2 \)  
5. \( r_2; r_1; w_2; w_1 \)  
6. \( r_2; w_2; r_1; w_1 \).

In general, we get \((2N + 1)! \div 2^N\) interleavings for \( T_0 \parallel T_1 \parallel \ldots \parallel T_N \).
Concurrency Problem Encoding

We define the following preserved-program order (PPO) for the problem challenge $T_0 \parallel T_1 \parallel \ldots \parallel T_N$:

Let $R \triangleq \{r_0, \ldots, r_N\}$ and $W \triangleq \{w_{init}, w_0, \ldots, w_N\}$.

Our partial-order encodings are parameterized by three theories:

- $\mathcal{T}_C$: clock constraints, e.g. $c_r < c_w$ for $r \in R$ and $w \in W$,
- $\mathcal{T}_S$: selection constraints, e.g. $s_r = s_w$ for $r \in R$ and $w \in W$,
- $\mathcal{T}_V$: read constraints, e.g. $rv_r$ is a unique $\mathcal{T}_V$-variable for $r \in R$. 
Cubic-size Encoding of Concurrency Problem

Let $\phi^3$ be the $O(N^3)$ partial-order encoding of $T_0 \parallel T_1 \parallel \ldots \parallel T_N$:

\[
\begin{align*}
C_{W_{init}} &< C_{r_{assert}} \land \bigwedge_{i=1,\ldots,N} C_{W_{init}} < C_{r_i} < C_{W_i} \land \bigwedge_{w,w'\in W, w\neq w'} (C_{w} < C_{w'} \vee C_{w'} < C_{w}) \land S_{w} \neq S_{w'} \land \\
\bigwedge_{w\in W, r\in R} (C_w < C_r \vee C_r < C_w) \land \bigwedge_{w\in W} S_w = S_r \land r_{v_{r_{assert}}} > N \land \\
\bigwedge_{w\in W, r\in R} (S_w = S_r) \Rightarrow C_w < C_r \land \bigwedge_{r\in R} (S_{w_{init}} = S_r) \Rightarrow 0 = r_{v_r} \land \bigwedge_{i=1,\ldots,N,r\in R} (S_{w_i} = S_r) \Rightarrow r_{v_{r_i + 1}} = r_{v_r} \land \\
\bigwedge_{w,w'\in W, r\in R} (S_w = S_r \land C_w < C_{w'}) \Rightarrow C_r < C_{w'} \land \\
\bigwedge_{w\in W, r\in R} (S_w = S_r) \Rightarrow r_{v_{r_{init}}} = 0 \land \bigwedge_{r\in R} (S_{w_{assert}} = S_r) \Rightarrow r_{v_r} + 1 = r_{v_{r_{assert}}} \land \\
\end{align*}
\]

**PPO**

**WW[x]**

**RW[x]**

**RF_{TO}[x]**

**assert(v_0 \leq N)**

**RF^{3}[x]**

**FR[x]**
Factorial Lower Bound DPLL(\(T\)) Proof Complexity

**Theorem (Lower Bound for Cubic Partial-Order Encoding)**

All Fixed-Alphabet-DPLL(\(T\)) proofs for the problem challenge encoded with \(\phi^3\) contain at least \(N!\) applications of \(T\)-\textsc{learn}.

We also have a quadratic-size partial-order encoding.\(^2\)

This asymptotically smaller encoding has also at least factorial-sized DPLL(\(T\)) proofs for our challenge problem!

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Experiments with Two Partial-Order Encodings

$E^3$ and $E^2$ are partial-order encodings of asymptotically different size, parameterized by three theories $\mathcal{T}_C$, $\mathcal{T}_S$ and $\mathcal{T}_V$.

We instantiate $\langle \mathcal{T}_C, \mathcal{T}_S, \mathcal{T}_V \rangle$ to four configurations:

1. “real-clk-int-val”: $\mathcal{T}_C = \mathcal{T}_S = \mathcal{T}_R$ and $\mathcal{T}_V = \mathcal{T}_Z$
2. “bv-clk-int-val”: $\mathcal{T}_C = \mathcal{T}_S = \mathcal{T}_{BV}$ and $\mathcal{T}_V = \mathcal{T}_Z$
3. “real-clk-bv-val”: $\mathcal{T}_C = \mathcal{T}_S = \mathcal{T}_R$ and $\mathcal{T}_V = \mathcal{T}_{BV}$
4. “bv-clk-bv-val”: $\mathcal{T}_C = \mathcal{T}_S = \mathcal{T}_{BV}$ and $\mathcal{T}_V = \mathcal{T}_{BV}$

We use the following SMT solvers: Boolector, CVC4, Yices2, Z3.

Example: “z3-bv-clk-int-val-$E^2$” denotes experiments with the $O(N^2)$ encoding using Z3 where $\mathcal{T}_C = \mathcal{T}_S = \mathcal{T}_{BV}$ and $\mathcal{T}_V = \mathcal{T}_Z$. We have a total of 56 SMT-LIB benchmarks. Timeout is 1 hour.
Experimental Results

Factorial growth of conflicts in `fkp2013-unsat` benchmark.
Concluding Remarks

- We have studied a simple, yet challenging concurrency problem.
- Our experiments provide an important diagnostic practice in the development of SMT encodings.
- The proofs we have manually inspected in CVC4 pinpoint value constraints as culprits.
- This way, our experiments can guide research into improving the performance of SMT solvers on such benchmarks.

The results of our work will shortly be published in SMT’15.
Concluding Remarks

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Thank you!
SC-relaxed Consistency Encoding

Let $E$ be the set of events, $\ll$ be the PPO, $\text{val} : E \to \mathcal{T}_V$-terms, $\text{guard} : E \to \mathcal{T}_V$-formulas and $L$ be the set of memory locations.

$$
PPO \triangleq \bigwedge \{ (\text{guard}(e) \land \text{guard}(e')) \Rightarrow (c_e < c_{e'}) \mid e, e' \in E : e \ll e' \}$$

$$
WW[x] \triangleq \bigwedge \{ (c_w < c_{w'} \lor c_{w'} < c_w) \land s_w \neq s_{w'} \mid w, w' \in W_x \land w \neq w' \}$$

$$
RW[x] \triangleq \bigwedge \{ c_w < c_r \lor c_r < c_w \mid w \in W_x \land r \in R_x \}$$

$$
\text{RF}_{TO}[x] \triangleq \bigwedge \{ \text{guard}(r) \Rightarrow \lor \{ s_w = s_r \mid w \in W_x \} \mid r \in R_x \}$$

$$
\text{RF}^3[x] \triangleq \bigwedge \{ (s_w = s_r) \Rightarrow (\text{guard}(w) \land \text{val}(w) = rv_r \land c_w < c_r) \mid r \in R_x \land w \in W_x \}$$

$$
\text{FR}[x] \triangleq \bigwedge \{ (s_w = s_r \land c_w < c_{w'} \land \text{guard}(w')) \Rightarrow (c_r < c_{w'}) \mid w, w' \in W_x \land r \in R_x \}$$

$$
\mathcal{E}^3 \triangleq \bigwedge \{ \text{RF}_{TO}[x] \land \text{RF}^3[x] \land \text{FR}[x] \land WW[x] \land RW[x] \mid x \in L \} \land \text{PPO}$$

$$
\text{RF}^2[x] \triangleq \bigwedge \{ (s_w = s_r) \Rightarrow (c_w = \text{sup}_r \land \text{guard}(w) \land \text{val}(w) = rv_r \land c_w < c_r) \mid r \in R_x \land w \in W_x \}$$

$$
\text{SUP}[x] \triangleq \bigwedge \{ (c_w \leq c_r \land \text{guard}(w)) \Rightarrow (c_w \leq \text{sup}_r) \mid r \in R_x \land w \in W_x \}$$

$$
\mathcal{E}^2 \triangleq \bigwedge \{ \text{RF}_{TO}[x] \land \text{RF}^2[x] \land \text{SUP}[x] \land WW[x] \land RW[x] \mid x \in L \} \land \text{PPO}$$